6.1 Chebyshev Polynomials

Recall that the error in polynomial interp. is given by

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

- Assume (for convenience) that the interpolation nodes are in $[-1,1]$. If $x \in [-1,1]$ then $\xi_x \in [-1,1]$, so

$$\max_{x \in [-1,1]} |f(x) - p(x)| \leq \frac{1}{(n+1)!} \max_{x \in [-1,1]} |f^{(n+1)}(\xi_x)| \cdot \max_{x \in [-1,1]} \left| \prod_{i=0}^{n} (x - x_i) \right|$$

Idea: Choose the nodes x_i to minimize this term!

Observe: $\prod_{i=0}^{n}(x-x_i)$ is a monic polynomial

coefficient of x^n is 1.

Theorem: If p is a monic polynomial of degree n then

$$\|p\|_\infty := \max_{x\in[-1,1]} |p(x)| \geq 2^{1-n} \quad (*)$$

To prove this, we will construct a polynomial that achieves the bound.

Chebyshev Polynomials:

2 Definitions (equivalent)

Recursive Definition:

$$\begin{cases} T_0(x) = 1, \quad T_1(x) = x \\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1 \end{cases}$$

$\Rightarrow T_2(x) = 2x^2 - 1$

$T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$ ...

$\vdots$

Equivalent Definition / Theorem

$T_n(x) = \cos(n \cos^{-1} x)$, $n \geq 0$

proof of equivalence:

Since $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\Rightarrow \cos(n+1)\theta = \cos\theta \cos n\theta - \sin\theta \sin n\theta$

$\cos(n-1)\theta = \cos\theta \cos n\theta + \sin\theta \sin n\theta$

$\Rightarrow \cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta \cos n\theta$

↑ plug in $\cos^{-1} x$

& define $f_n(x) = \cos(n \cos^{-1} x) \Rightarrow f_0(x) = 1,\ f_1(x) = x$

& $f_{n+1}(x) + f_{n-1}(x) = 2x f_n(x)$

$\Leftrightarrow f_{n+1}(x) = 2x f_n(x) - f_{n-1}(x)$

So $T_n = f_n \quad \forall n$ ■

Recall that we want to get an upper bound on $|p(x)|$, $x \in [-1,1]$ when p is monic.

Theorem: $\|p\|_\infty = \max_{x \in [-1,1]} |p(x)| \geq 2^{1-n}$

proof: Suppose that $|p(x)| < 2^{1-n} \quad \forall x \in [-1,1]$

We want to get a contradiction.

Let $q(x) = 2^{1-n} T_n(x)$ & let $x_i = \cos\left(\frac{i\pi}{n}\right)$.

Observe that q is monic, degree n. Also:

$$q(x_i) = 2^{1-n} \cos\left(n \cdot \frac{i\pi}{n}\right) = 2^{1-n}(-1)^i$$

So $q(x_i)(-1)^i = 2^{1-n} > |p(x_i)| \geq (-1)^i p(x_i)$

by our supposition

$$\Rightarrow (-1)^i \big(\underbrace{\overset{\text{monic}}{q(x_i)} - \overset{\text{monic}}{p(x_i)}}_{\text{poly., degree} \leq n-1 \text{ (highest degree terms cancel)}}\big) > 0 \quad \forall i \in \{0, \dots, n\}$$

So, we have a polynomial of degree $\leq n-1$ that changes sign $n+1$ times in $[-1,1] \Rightarrow$ it has n-roots. Can't happen with degree $\leq n-1$ $\Rightarrow$ contradiction $\Rightarrow$ $|P_n(x)| \geq 2^{1-n}$ ∎

OK, let's get back to the error in poly. interp.

$$\max_{x\in[-1,1]} |f(x)-p(x)| \leq$$

$$\frac{1}{(n+1)!} \max_{x\in[-1,1]} |f^{(n+1)}(x)| \max_{x\in[-1,1]} \underbrace{\left|\prod_{i=0}^{n}(x-x_i)\right|}_{\text{monic degree } n}$$

$$\Rightarrow \geq 2^{-n}$$

So, the best we can hope to do is 2^{-n}.

From the proof above, we want $\prod_{i=0}^{n}(x-x_i) = 2^{-n}T_{n+1}(x)$

with nodes $x_i = \cos\left(\frac{2i+1}{2n+2}\pi\right)$, $i=0,\cdots,n$

So we proved:

Theorem

If x_i are the the roots of T_{n+1}:

$$|f(x) - p(x)| \leq \frac{2^{-n}}{(n+1)!} \max_{|t| \leq 1} |f^{(n+1)}(t)|$$